Mathematical Background

This document contains summary of some of the items discussed as part of the COMS 3110 Lectures. This document **does not replace the lecture materials**. This document may contain some topics that are not covered as part of the lecture; you will not be tested on those parts, they are made available to you for gaining further knowledge on topics/concepts that are related to class-lecture.

1 Topics Covered

- Summations
- Sets
- Proof techniques

2 Series

Finite series. You should be able to prove-by-induction the following statements. Try it out to improve your skills on application of induction proofs.

$$1 + 2 + 3 + \ldots + n = n(n+1)/2$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2(n+1)^2/2^2$$

$$1 + r + r^2 + r^3 + \ldots + r^{n-1} = (r^n - 1)/(r - 1)$$

Infinite series.

$$1 + r + r^2 + r^3 + \ldots = 1/(1-r) \text{ when } r < 1$$

3 Semantics of Summation

Assume: $L \leq U$:

$$\sum_{i=L}^{U} f(i) = f(L) + f(L+1) + \ldots + f(U)$$

$$\sum_{i=L}^{U} f = \underbrace{f + f + f + \ldots + f}_{(U-L+1) \text{ occurrences of } f}$$

4 Sets

Please refer to section B.1 in the textbook.

5 Proof by Contradiction

Let's P be the statement we want to prove. The idea is to show that assuming the opposite of P leads to a logical inconsistency.

1. Assume the negation of the statement you want to prove.

2. Derive the contradiction.
3. This contradiction then negates the assumption.

6 Proof by Induction

In proofs by induction, the statement we want to establish usually depends on an integer parameter. Let $P(n)$ be the statement that you want to prove is true for all integers $n \geq n_0$, where n_0 is some fixed starting point. The idea is to first prove that $P(n_0)$ holds and then show that if $P(k)$ holds for an arbitrary k, then $P(k+1)$ must hold.

We usually follow the following template when using induction.

- **Step 1: Base case** Prove that the statement is true for the initial value $n = n_0$.
- **Step 2: Induction hypothesis** Assume that the statement holds for some arbitrary positive integer $n = k$.
- **Step 3: Induction step** Using the inductive hypothesis, prove that the statement also holds for $n = k + 1$.

This allows us to conclude that the statement holds for all integers $n \geq n_0$.

Remark: In Step 2, we assume that the statement holds for $n = k$. This is known as weak induction. A variation, called strong induction, modifies Step 2 to assume that the statement holds for all n such that $n_0 \leq n \leq k$.

7 Exercise

1. Prove that $\sum_{i=1}^{n} i \times i! = (n+1)! - 1$
2. Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$