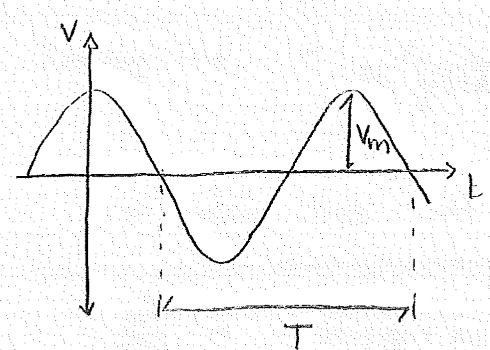
Sinusoidal Steady State Analysis:

So far we have been discussing circuits with constant sources.

We now turn our attention to circuits that have time-varying sources.

In particular we are interested in circuits that utilize sinusoidal sources.

The Sinusoidal Source:



A sinusoidal source produces a voltage or current that varies sinusoidally with time.

We will assume that our sources are defined by a cos function.

$$v(t) = V_m \cos(\omega t + \phi)$$

There are 4 important characteristics

1) The period of the wave form: T
   This is how long it takes for the waveform to repeat.

2) The frequency:

$$f = \frac{1}{T} \quad [Hz]$$

or

$$\omega = 2\pi f \quad [rad/s]$$

3) Amplitude V_m or I_m

4) Phase angle ϕ: This defines the starting point of the waveform (i.e. the value of the function at $t=0$)

(Some pages are from Prof. Nathan Neihart, another 201 instructor)

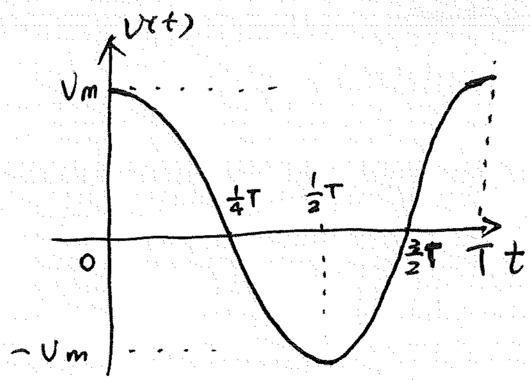
$v(t) = V_m \cos(\omega t + \phi)$

- ϕ → initial phase
- ω → periodic signal, with frequency : $f = \frac{\omega}{2\pi}$

  period : $T = \frac{2\pi}{\omega}$

  ω: converting f or T into phase
- V_m → Amplitude.

$\pi = \omega t = 2\pi \cdot \frac{\frac{1}{2}T}{T} = \pi.$



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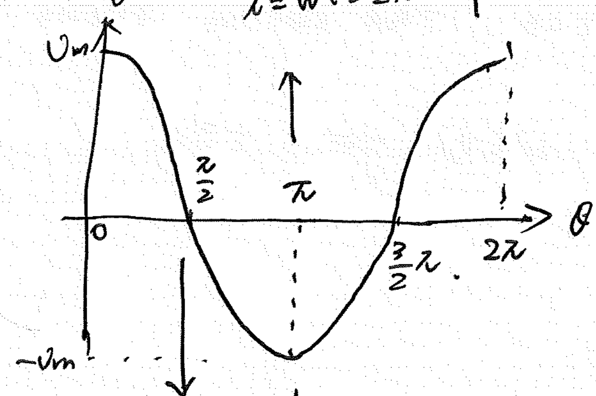


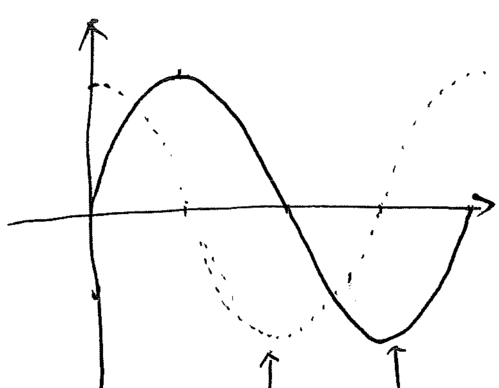
$V_m \cos \omega t$

$\frac{\pi}{2} = \omega t = 2\pi \cdot \frac{\frac{1}{4}T}{T} = \frac{1}{2}\pi.$

It is very important to convert time domian in to radian degrees (phase)

t θ

shifted : $\frac{T}{4}$, or $\frac{\pi}{2}$.



~~shifted~~

shifted $\frac{T}{4}$, convert to :

$$2\pi \cdot \frac{\frac{T}{4}}{T} = \frac{\pi}{2}.$$

* The unit must match !!

$\cos \omega t$ $\cos(\omega t - \frac{\pi}{2})$ ← radian.

$(\sin \omega t)$

or $\cos(\omega t - 90°)$ ← degree ? $\frac{90°}{360°} \times 2\pi \Leftrightarrow$ radian

* The units of "ωt" and "ϕ" must match before the function can be correctly evaluated.

It is also useful to know the RMS value of the sinusoidal source:

RMS → Root Mean Squared

For any continuous function defined over the interval $T_1 \le t \le T_2$ the RMS value is given by:

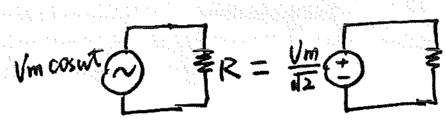
$$X_{rms} = \sqrt{\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} [X(t)]^2 \, dt} \quad \leftarrow \text{general form.}$$

For our sinusoidal source the RMS value is:

$$V_{RMS} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi) \, dt} = \frac{V_m}{\sqrt{2}}$$

$$\boxed{V_{rms} = \frac{V_m}{\sqrt{2}}} \text{ for sinusoidal signals}$$

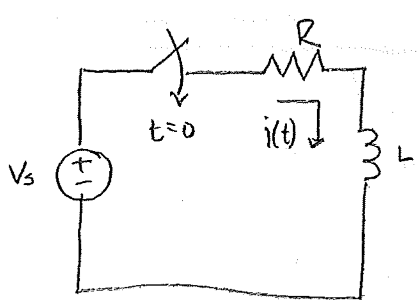
This will be particularly important when we begin discussing power.



interms of Power!

The sinusoidal response

Consider:



Assume $V_s = V_m \cos(\omega t + \phi)$

If we want to find an expression for $i(t)$ we can KVL:

$$L \frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

C.

The solution this differential equation is:

$$i(t) = \underbrace{\frac{-V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\phi-\theta)\, e^{-\frac{R}{L}t}}_{\text{Transient Response}} + \underbrace{\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t+\phi-\theta)}_{\text{Steady-State Response.}}$$

θ is defined to be: $\tan(\theta) = \frac{\omega L}{R}$

The first term is referred to as the transient response.

→ It is scaled by $e^{-R/L\, t}$ and hence will approach zero as time elapses.

The second term is referred to as the steady-state response.

→ It exists as long as the switch is closed and as long as the source supplies a sinusoidal voltage.

→ We are interested in developing a technique to solve for the steady-state response directly.

The advantage of doing this is that we don't have to solve the differential equation.

The disadvantage is that we cannot obtain the transient response. By extension we cannot find the total response.

This is generally okay because many systems spend most of their time in the steady state.

The steady-state solution has the following characteristics.

1) The steady-state solution is a sinusoidal function.

2) The frequency of the steady-state response signal is identical to the source signal. This is always true for linear circuits with <u>constant</u> parameters.

3) The maximum amplitude of the steady-state response is, in general, different from that of the source.

4) The phase angle of the response is, in general, different than that of the source.

Why do we mention these characteristics?

* Notice that once we decide to find only the steady-state response, the task is simplified to finding the $\boxed{\text{amplitude}}$ and $\boxed{\text{phase angle}}$ of the steady-state response. The waveform and frequency are already known (from the source).

The Phasor:

Since we are only interested in the steady state response, our task involves finding the amplitude and phase angle of the response.

A useful tool that can be used in this case is the Phasor:

Consider Euler's Identity: $$e^{\pm j\theta} = \cos(\theta) \pm j\sin(\theta)$$

We can think of the cosine as the real part of the above equation and the sin as the imaginary part:

$$\cos\theta = \mathcal{R}\{e^{j\theta}\} \qquad \sin(\theta) = \mathcal{I}\{e^{j\theta}\}$$

or $\quad Re\{e^{j\theta}\} \qquad Im\{e^{j\theta}\}$

Remember that we have already chosen to use the cosine function in our analysis.

$$V = V_m\cos(\omega t + \phi)$$
$$= V_m \mathcal{R}\{e^{j(\omega t + \phi)}\} = V_m \mathcal{R}\{e^{j\omega t}e^{j\phi}\}$$

Finally we can express the voltage as:

$$V = \mathcal{R}\{V_m e^{j\phi} e^{j\omega t}\}$$

Notice that $V_m e^{j\phi}$ is a complex number that carries the amplitude and phase angle of a given sinusoidal function.

This complex number is, by definition, the phasor representation of a given sinusoidal function.

The phasor transform is:

$$\boxed{\vec{V} = \underline{\underline{V_m}}\, e^{j\underline{\underline{\phi}}} = P\{V_m \cos(\omega t + \phi)\}}$$

Amplitude phase

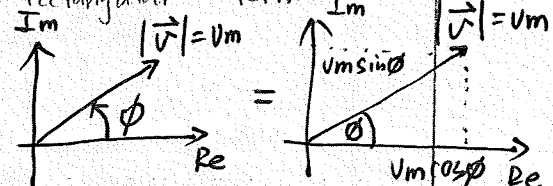
We throw away the "ωt". Because, in a Linear System the frequency information won't change. It will not generate new frequencies.

$P\{\cdot\}$ denotes the phasor transform.

Notice that the phasor transform takes a sinusoidal function from the time domain (i.e. a function of time) and transforms it to a complex number in the frequency domain (i.e. a function of frequency ω).

We will typically use the polar form to represent phasors, but we can also represent phasors using rectangular form:

$$\underline{\vec{V} = V_m e^{j\phi} = V_m \cos\phi + j V_m \sin\phi}$$



Because of the frequent use of $e^{j\phi}$ a special notation has been developed:

$$V_m e^{j\phi} = V_m \angle \phi^\circ \quad \leftarrow \text{referred to as angle notation}$$

We can also define the inverse phasor transform to take a complex number in the frequency domain to a sinusoidal function in the time domain.

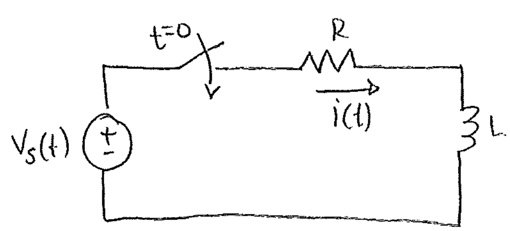
$$\boxed{P^{-1}\{V_m e^{j\phi}\} = \mathcal{R}\{V_m e^{j\phi} e^{j\omega t}\}}$$

"putting back ωt information"

Ex: $\vec{V} = 100\angle -26^\circ$ $\qquad P^{-1}\{\vec{V}\} = 100\cos(\omega t - 26^\circ)$ ← degree!

Notice that we cannot know anything about the frequency, ω, because the phasor only carries information about amplitude and phase angle.

Now lets see how this new tool can be used in finding the steady-state response of our initial circuit!



$$V_s(t) = V_m \cos(\omega t + \phi)$$